A NEW CANONICAL FORM OF THE ELLIPTIC INTEGRAL*

BY

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Introduction

The elliptic norm curve Q_n in space S_{n-1} admits a group G_{2n^2} of collineations, and in fact there is a single infinity of such curves which admit the same group. A particular Q_n of the family is distinguished by a value of the parameter τ, itself an elliptic modular function defined by the modular group congruent to identity (mod. n).

In the group G_{2n^2}, there are certain involutory collineations with two fixed spaces. If Q_n is projected from one fixed space upon the other, a family of rational curves R_m mapping the family of Q_n's is obtained. The quadratic irrationality separating involutory pairs on Q_n involves the modulus τ, and the parameter t of the R_m. When the genus of the modular group is zero and $n = 3, 4, 5$, the irrationality can be used to define the elliptic parameter

$$u = \int \frac{(t\,dt)}{\sqrt{(t\tau)\,\alpha_\tau^{r-3}\,\alpha_t^3}},$$

where α_t^r is the tetrahedral, octahedral, or icosahedral form. This is in contrast with Klein's form as developed by Bianchi, for there the normal elliptic integral is a rational curvilinear integral taken along an elliptic curve.

A comparison of the two integrals is more illuminating if carried out for a special case. Let Q_n be Q_5 in S_4. Let $\phi_i(x_j, a) = 0$, $(i, j = 0, \cdots 4)$ be five quadrics intersecting in Q_5, where according to Bianchi's notation x_j are the variables, a the modulus. After a suitable transformation of coördinates the icosahedral form which appears in the irrationality is

$$\alpha_t^{12} = t_1\, t_2\,(t_1^{10} + 11 t_1^5\, t_2^5 - t_2^{10}).$$

The integral u involving $\tau \equiv a$ explicitly in a rather simple form is uniquely defined. Moreover it is invariant under all cogredient transformations of t and τ, which leave α_x^{12} unaltered, i. e., the sixty transformations of the icosahedral group applied simultaneously to t and τ, the parameter of the doubly covered conic R_2 and the modulus of Q_5 respectively, leave u unaltered.

Consider now Bianchi's integral, defined as

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$$U = C \int \frac{(w\, dv - v\, dw)}{(\phi_0 \phi_1 \phi_2 w v)},$$

where C is a constant, w and v any two expressions linear in x_j, and the denominator is the functional determinant of ϕ_0, ϕ_1, ϕ_2, w, v. Different expressions for U can be obtained by making different choices for w and v. Hence there is no unique form for U. Under the G_{50} of collineations on x_j and under the transformations of $a \equiv \tau$, U assumes various conjugate forms.

Hence u has the advantage over U in simplicity and uniqueness of form, and also in its invariancy under transformations.

In section I by a study of the integral u some interesting results are obtained. The modular equation connecting τ and J, the absolute invariant of u, can be deduced as the result of the binary syzygy of lowest weight connecting the concomitants of α_x^r. The requirement that the Riemann surface attached to the modular equation be regular leads to the modular equations associated with the regular bodies. It is then possible to eliminate the more tedious individual proofs used by Bianchi in the discussion of the moduli of Q_3 and Q_5 to show that these moduli are the tetrahedral and icosahedral irrationalities respectively. In fact the algebraic discussion carried out once for α_x^r is complete for factor groups of genus zero, which have been discussed by Klein, i. e., those isomorphic with the groups associated with the regular bodies, namely the one dihedral group G_6, and the tetrahedral, octahedral, and icosahedral groups.

In sections II, III, IV, it is shown that the canonical form u occurs naturally in connection with the elliptic norm curves Q_3, Q_4, Q_5 and in section III the possibility of extending the canonical form to larger values of the genus is indicated.

Whenever possible the notation conforms to that of Klein-Fricke, Bianchi, and Grace and Young.*

I. A SYZYGY

1. Consider the elliptic integral of the first kind in the form given in the introduction.† The form under the radical sign, since it is a quartic in t,

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* Abbreviations and references:

K. F. *Vorlesungen über die Theorie der elliptischen Modulfunctionen,* by Klein and Fricke.

B. *Über die Modulformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung,* by Bianchi in Mathematische Annalen, vol. 17 (1880), pp. 234–262.

G. & Y. *Algebra of Invariants,* by Grace and Young.

† The integral u can be reduced to Weierstrass's canonical form by means of the transformation $x_1 = \alpha_\tau^{n-1}\alpha_t$, $x_2 = (t\tau)$. (Vide Clebsch, *Theorie der binären algebraischen Formen,* §§ 81–83, for similar transformations.)

has two invariants* g_2 and g_3, the former of degree $2(n-2)$ in τ, the latter of degree $3(n-2)$ in τ. But $f \equiv \alpha_\tau^n$ has one covariant, H, of degree $2(n-2)$ in τ, and one covariant, $(f, H)^1$, of degree $3(n-2)$ in τ. Hence g_2 must be some multiple of H; g_3 some multiple of $(f, H)^1$. The discriminant, Δ, of a quartic can be expressed in terms of g_2^3 and g_3^2. The absolute invariant J is then defined by the syzygy

$$J : J - 1 : 1 :: g_2^3 : 27g_3^2 : \Delta.$$

In order to obtain an analogous syzygy involving the transvectants of f, it is necessary to obtain the expressions for g_2, g_3, and Δ, in terms of the transvectants of f and to determine some relation between these forms.

2. Let

$$f \equiv \alpha_\tau^n \equiv \beta_\tau^n \equiv \gamma_\tau^n \equiv \text{etc.},$$

$$g \equiv \alpha_\tau^{n-3}\,\alpha_t^3\,(t\tau) \equiv \beta_\tau^{n-3}\,\beta_t^3\,(t\tau) \equiv \text{etc.},$$

$$H \equiv (\alpha\beta)^2\,\alpha_\tau^{n-2}\,\beta_\tau^{n-2} = (f,f)^2,$$

$$H_g \equiv (g,g)^2,$$

$$\tilde{t} \equiv (\alpha\beta)^2(\gamma\alpha)\,\alpha_\tau^{n-3}\,\beta_\tau^{n-2}\,\gamma_\tau^{n-1} = (f, H)^1,$$

$$i \equiv (\alpha\beta)^4\,\alpha_\tau^{n-4}\,\beta_\tau^{n-4} = (f,f)^4,$$

$$j \equiv (\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2\,\alpha_\tau^{n-4}\,\beta_\tau^{n-4}\,\gamma_\tau^{n-4} = (f, i)^2,$$

$$k \equiv (\alpha\beta)^6\,\alpha_\tau^{n-6}\,\beta_\tau^{n-6} = (f,f)^6.$$

According to Klein $(g, g)^4$ equals $2g_2$ and $(H_g, g)^4$ equals $4g_3$.

$$(g,g)^4 = \alpha_\tau^{n-3}\,\beta_\tau^{n-3}\big(\alpha_\tau^3(t\tau),\,\beta_\tau^3(t\tau)\big)^1 = -\frac{18}{4!}(\alpha\beta)^2\,\alpha_\tau^{n-2}\,\beta_\tau^{n-2}.$$

$$\therefore\ g_2 = -\tfrac{3}{8}H.$$

$$H_g\dagger = \tfrac{1}{2}(\alpha\beta)^2\,\alpha_\tau^{n-3}\,\beta_\tau^{n-3}\,\alpha_t\,\beta_t\,(t\tau)^2,$$

$$(H_g, g)^4 = \frac{6}{4!}(\alpha\beta)^2(\gamma\alpha)\,\alpha_\tau^{n-3}\,\beta_\tau^{n-2}\,\gamma_\tau^{n-1}.$$

$$\therefore\ g_3 = \tfrac{1}{16}(f, H)^1 = \tfrac{1}{16}\tilde{t}.$$

Now $\tilde{t}$ is a jacobian. Therefore its square‡ is reducible. In fact

$$-2\tilde{t}^2 - H^3 = (H,H)^2 f^2 - 2(f,H)^2 fH. \tag{1}$$

Let $\Delta' = 2\tilde{t}^2 + H^3$ and $J = H^3/\Delta'$. Then $J - 1 = -2\tilde{t}^2/\Delta'$ and the following proportion can be written

* K. F., I, pp. 13–15.

† G. & Y., §§ 43, 49, for methods of calculating transvectants.

‡ G. & Y., § 78.

$$J : J - 1 : 1 = H^3 : -2\tilde{t}^2 : \Delta'. \tag{2}$$

This is assuredly a relation between the hessian and jacobian of an $n$ic, similar to the one existing between the invariants, g_2 and g_3, of a quartic. Corresponding to the discriminant Δ of the quartic is the form Δ', expressed by (1) in terms of transvectants which are reducible. Hence before (2) is discussed, the reduction of $(H, H)^2$ and of $(f, H)^2$ is made.

3. The reduction just mentioned can be outlined as follows:

$$\begin{aligned}(f, H)^2 &= \frac{1}{2n-5}(\alpha\beta)^2\{(n-3)(\alpha\gamma)^2\alpha_\tau^{n-4}\beta_\tau^{n-2}\gamma_\tau^{n-2} \\ &\quad + (n-2)(\alpha\gamma)(\beta\gamma)\alpha_\tau^{n-3}\beta_\tau^{n-3}\gamma_\tau^{n-2}\}^* \\ &= (\alpha\beta)^2(\alpha\gamma)^2\alpha_\tau^{n-4}\beta_\tau^{n-2}\gamma_\tau^{n-2} - \frac{n-2}{2(2n-5)}if \\ &= (\alpha\beta)^2(\alpha\gamma)^2\beta_\tau^2\gamma_\tau^2\alpha_\tau^{n-4}\beta_\tau^{n-4}\gamma_\tau^{n-4} - \frac{n-2}{2(2n-5)}if^\dagger \\ &= \tfrac{1}{2}if - 2(\beta\gamma)^2(\alpha\beta)(\alpha\gamma)\alpha_\tau^{n-2}\beta_\tau^{n-3}\gamma_\tau^{n-3} - \frac{n-2}{2(2n-5)}if^\ddagger \\ &= \frac{n-3}{2(2n-5)}if.\end{aligned}$$

The reduction of $(H, H)^2$ is accomplished by means of Gordan's series.§ Since $(H, H)^2 = ((f,f)^2, H)^2$, the first series used is

$$\begin{pmatrix} f & f & H \\ n & n & 2n-4 \\ 0 & 2 & 2 \end{pmatrix}.$$

This gives

$$\begin{aligned}(H, H)^2 + \frac{n-2}{2(2n-5)}iH &= ((f, H)^2, f)^2 + \frac{4(n-3)}{3n-8}((f, H)^3, f)^1 \\ &\quad + \frac{2(2n-7)}{3(3n-10)}((f, H)^4, f)^0.\end{aligned} \tag{3}$$

The terms on the right-hand side of (3) must be reduced in turn so that (3) may contain only the transvectants enumerated in § 2. It is found that

* G. & Y., § 22. Apply identity III.

† Square identity III (loc. cit.) and apply it.

‡ G. & Y., § 22. Rewrite identity I thus: $(\beta\gamma)\alpha_\tau = (\beta\alpha)\gamma_\tau - (\gamma\alpha)\beta_\tau$ and multiply by $(\beta\gamma)(\alpha\beta)(\alpha\gamma)$ before applying.

§ G. & Y., § 54.

$$((f, H)^2, f)^2 = \frac{1}{2(2n-5)(3n-8)}\left\{n(n-3)iH - \frac{(2n-8)(2n-9)}{3}jf + \frac{(n-4)(n-5)}{3}kf^2\right\}.$$

The transvectant $(f, H)^3$ is given by the series

$$\begin{pmatrix} f & f & f \\ n & n & n \\ 0 & 2 & 3 \end{pmatrix},$$

and equals

$$\frac{n-4}{2(2n-5)}(f, i)^1,$$

$$((f, H)^3, f)^1 = \left(\frac{n-4}{2(2n-5)}(f, i)^1, f\right)^{1*}$$

$$= \frac{n-4}{2(2n-5)}\left\{-\frac{2n-9}{3n-10}\left[-j + \frac{n-5}{2(2n-9)}kf\right] + \tfrac{1}{2}iH\right\}.$$

The transvectant $(f, H)^4$ is given by the series

$$\begin{pmatrix} f & f & f \\ n & n & n \\ 0 & 4 & 2 \end{pmatrix}.$$

$$\therefore (f, H)^4 = -\frac{n-1}{2n-5}(i, f)^2 + \frac{(3n-10)(n-5)}{4(2n-7)(2n-9)}kf$$

$$= \frac{n-1}{2n-5}j + \frac{(n-4)(n-5)}{4(2n-7)(2n-9)}kf.$$

If the values of the different transvectants are substituted in (3), the following result is obtained:

$$(H, H)^2 = -\frac{1}{2(2n-5)}iH + \tfrac{1}{3}jf.$$

It is interesting to notice that the term kf^2 has the coefficient zero. Therefore k, a form which does not exist in the case of the quartic, does not appear in the final result.

4. The relation between t and H is given by (1). If now the results of §§ 2 and 3 are used, (1) can be replaced by

$$2t^3 + H^2 = f^2(\tfrac{1}{2}iH - \tfrac{1}{3}jf).$$

Then (2) can be written as follows:

* See G. & Y., § 77 (XIX), for the formula used to obtain the result given above.

(4) $$J : J - 1 : 1 = 6H^3 : -12\tilde{t}^2 : f^2(3iH - 2jf).$$

If the equivalents of H and $\tilde{t}$ (vide § 2) are introduced in (4), then the usual form of the syzygy existing between the invariants of the quartic is obtained. Thus (4) is the desired syzygy between the hessian and jacobian of an $n$ic. It shows that, as H and $\tilde{t}$ correspond to g_2 and g_3 respectively, Δ' corresponds to Δ.

5. The syzygy (4) containing the parameter τ and the invariant J is for a given J an algebraic equation of degree $6(n-2)$, with roots which are algebraic functions of J. It can be mapped on a $6(n-2)$-sheeted Riemann surface. The k-fold roots of (4) and the values of J giving the branch points of the surface must be determined.

Since a k-fold root of (4) is a $(k-1)$-fold root of the transvectant $(H^3, \tilde{t}^2)^1$, the reduction of this transvectant is necessary. Let

$$H \equiv (\alpha\beta)^2 \alpha_\tau^{n-2} \beta_\tau^{n-2} \equiv h_\tau^{2(n-2)},$$

$$\tilde{t} \equiv (\delta\epsilon)^2 (\eta\delta) \delta_\tau^{n-3} \epsilon_\tau^{n-2} \eta_\tau^{n-1} \equiv t_\tau^{3(n-2)}.$$

Then

$$(H^3, \tilde{t}^2)^1 = \frac{H^2\tilde{t}(\alpha\beta)^2(\delta\epsilon)^2(\eta\delta)}{6(n-2)} \alpha_\tau^{n-3}\beta_\tau^{n-3}\{(n-3)(\alpha\delta)\beta_\tau \delta_\tau^{n-4}\epsilon_\tau^{n-2}\eta_\tau^{n-1}$$
$$+ (n-2)(\alpha\epsilon)\beta_\tau \delta_\tau^{n-3}\epsilon_\tau^{n-3}\eta_\tau^{n-1} + (n-1)(\alpha\eta)\beta_\tau \delta_\tau^{n-3}\epsilon_\tau^{n-2}\eta_\tau^{n-2}$$
$$+ (n-3)(\beta\delta)\alpha_\tau \delta_\tau^{n-4}\epsilon_\tau^{n-2}\eta_\tau^{n-1} + (n-2)(\beta\epsilon)\alpha_\tau \delta_\tau^{n-3}\epsilon_\tau^{n-3}\eta_\tau^{n-1}$$
$$+ (n-1)(\beta\eta)\alpha_\tau \delta_\tau^{n-3}\epsilon_\tau^{n-2}\eta_\tau^{n-2}\} = H^2\tilde{t}(H, \tilde{t})^1$$
$$= H^2\tilde{t}f\left\{\frac{n-3}{3(n-2)}jf - \frac{2n-3}{3(n-2)}iH\right\}.$$

It is now evident that (4) has

$2(n-2)$ triple roots—the values for which $H = 0$;

$3(n-2)$ double roots—the values for which $\tilde{t} = 0$;

n double roots—the values for which $f = 0$;

$4(n-3)$ double roots—the values for which $(n-3)jf - (2n-3)iH = 0$.

The values of J giving the branch points of the Riemann surface are determined from the two following equations which are used instead of (4):

(4a) $$Jf^2(3iH - 2jf) = 6H^3,$$

(4b) $$(J-1)(3iH - 2jf)f^2 = -12\tilde{t}^2.$$

Then if

$$H = 0, \qquad J = 0;$$

$$\tilde{t} = 0, \qquad J = 1;$$

$$f = 0, \qquad J = \infty;$$

$$(n-3)jf - (2n-3)iH = 0,$$

$$J = \frac{6(3-n)H^2}{(n+3)if^2} = \frac{6(3-n)^3 j^2}{(n+3)(2n-3)^2 i^3}.$$

The form of the equations (4*a*) and (4*b*) shows that all the sheets of the $6(n-2)$-sheeted Riemann surface come together, k at a time, over the branch points $J = 0$ and $J = 1$ but do not do so over the remaining branch points.

6. The condition for the regularity of the Riemann surface at $J = \infty$ can be determined. The desirability of applying such a condition is obvious when the following theorem has been proved: *If the Riemann surface is regular over $J = \infty$, τ is an elliptic modular function.*

From (4) J is a uniform function of τ and of ω. By hypothesis the only critical points occur at $J = 0, 1, \infty$. But these give the vertices of the two nets of triangles obtained by mapping the real J axis on the τ and ω planes respectively.* The positive half of the J plane is mapped into a region within a triangle having one vertex at ∞ and the angles at the finite vertices equal to $\pi/2$ and $\pi/3$ respectively. This is true in both the τ and ω planes. Since a circuit around $J = 0$ necessitates $\omega(0) = \omega_0$† and $\tau(0) = \tau_0$‡ the two power series for J as a function of τ and of ω are

$$J = a_0(\omega - \omega_0)^3\{1 + P(\omega - \omega_0)\},$$

$$J = b_0(\tau - \tau_0)^3\{1 + P(\tau - \tau_0)\}.$$

Therefore τ is a uniform function of ω at ω_0; in fact

$$\tau - \tau_0 = \sqrt[3]{\frac{a_0}{b_0}}(\omega - \omega_0)\{1 + P(\omega - \omega_0)\}.$$

In like manner it is a uniform function of ω when $J = 1$. The first term of the series will be of the form

$$\tau - \tau_1 = \sqrt{\frac{a_1}{b_1}}(\omega - \omega_1),$$

where τ_1 and ω_1 are the values of τ and ω respectively for $J = 1$. A circuit around $J = \infty$ leads into the negative half-plane. This is not mapped in

* K. F., I, p. 45 ff.

† K. F., I, p. 128.

‡ (4).

the given region of the τ and ω plane but in the reflection of that region. Therefore in the map of the positive or negative half of the J plane, τ is a uniform function of ω and the theorem is proved.

But the condition that the Riemann surface be regular over $J = \infty$ is not yet known. In order to determine when all the sheets come together k at a time, assume that f and H have no common roots and hence that f has no double root. The form of $(H^3, \tilde{t})^1$ shows that $f = 0$ accounts for $2n$ sheets. Hence there are $6(n-2) - 2n = 4(n-3)$ sheets to be brought together over $J = \infty$ for the roots of $(n-3)jf - (2n-3)iH = 0$. But the corresponding form of J shows that only when i or f vanishes can $J = \infty$.

Let the values $(\mu_\tau^p)^k \equiv p^k = 0$ be those not included by $f = 0$, which give the branch point at $J = \infty$. The order of the branch point is then $k - 1$. Since the surface is to be regular

$$3iH - 2jf \backsim f^{k-2} p^k, * \tag{5}$$

and

$$(n-3)jf - (2n-3)iH \backsim f^{k-2} p^{k-1} q,$$

where q refers to any other possible branch points. Since p is to determine $J = \infty$,

$$i \backsim ph,$$

where h is the result of dividing i by p. Consider the order in the variables on each side of (5). Then

$$4(n-3) = n(k-2) + kp, \tag{6}$$

where p indicates the order of the form p. If $k = 2, p = 2(n-3)$. Therefore the form p cannot be a factor of i whose degree is $2(n-4)$, unless i vanishes identically. If $k = 3$, since H by hypothesis cannot contain f as a factor, i must do so, because of (5); i. e., $i \backsim fph'$ where h' is the result of dividing h by f. But when $k = 3$, from (6) $p = n - 4$. Hence

$$2(n-4) = n + n - 4 + h',$$

where h' indicates the order of the form h'. This is obviously impossible. In fact since iH contains a factor f^{k-2} (vide (5)), and H contains no factor f, i must contain the factor f^{k-2}. Therefore

$$2(n-4) = p + n(k-2) + h'.$$

If $k > 3$, $p + h' \lesseqgtr -8$, again an impossibility, unless $i \equiv 0$. If $i \equiv 0$, then from (4a) $J = \infty$ when $jf^3 = 0$. But in the case under discussion J can equal infinity only when i or f vanishes. Hence j is either a constant or some power, l, of f; i. e.,

* The symbol $\backsim$ is used to indicate equivalence of forms to within a multiplicative constant.

$$3(n-4) = ln$$

or

$$n(3-l) = 12.$$

If $l = 0, 1, 2$, in turn, $n = 4, 6, 12$. For the $n$ics, where n assumes only these values, the tetrahedral, octahedral, and icosahedral forms are obtained, when the condition $i \equiv 0$ is applied. On the other hand it is known* that in the case of the elliptic modular equations, the branch points of the Riemann surface can occur only at $J = 0, 1, \infty$.

The conclusions of the preceding paragraphs can be summarized as follows: *If the Riemann surface on which* (4) *is mapped is made regular, then* $i \equiv 0$, τ *is an elliptic modular function,* f *is a form associated with one of the regular bodies, and* (4) *is the corresponding modular equation.*

The dihedral case has not yet been shown to be included in (4) but can easily be developed. Let

$$f \equiv (\alpha_1 \tau_1 + \alpha_2 \tau_2)^n \equiv \tau_1^n - \tau_2^n.$$

Then

$$H = -2\tau_1^{n-2}\tau_2^{n-2},$$

$$\tilde{t} = -\tau_1^{n-3}\tau_2^{n-3}(\tau_1^n + \tau_2^n),$$

$$\dot{i} = -2\tau_1^{n-4}\tau_2^{n-4},$$

$$j \equiv 0.$$

The syzygy (4) becomes

$$J : J - 1 : 1 = -4\tau_1^n\tau_2^n : -(\tau_1^n + \tau_2^n)^2 : (\tau_1^n - \tau_2^n)^2.$$

This can easily be identified with the form of the dihedral equation given by Klein,† if the substitution $J = 1/z$ is made.

II. The elliptic norm curve of third order, Q_3, in two-dimensional space, S_2

7. Since the general configuration of the elliptic norm curve of third order Q_3 in two-dimensional space S_2, is familiar, it will be discussed in this paper only as it offers an opportunity of giving a geometric significance to some of the algebraic results previously obtained.

According to the work of Klein and Bianchi, Q_3, whose three homogeneous coördinates can be expressed as products of sigma functions, is transformed into itself by the group G_{18} of collineations which send the space S_2 into itself.‡ The normal form of its equation can be written

$$x_0^3 + x_1^3 + x_2^3 - 3\tau x_0 x_1 x_2 = 0. \tag{7}$$

* K. F., I, p. 574, § 1.

† Klein, *Vorlesungen über das Ikosaeder,* p. 60.

‡ K. F., II, pp. 246–256.

In the coördinate system used, the reference triangle is an inflexional triangle. Since τ is defined by (7) in terms of a ratio of the x's, τ is an elliptic modular function, i. e., a function of ω_1/ω_2, where ω_1 and ω_2 are the periods of the sigma functions in terms of which x_i is expressed.*

If in (7) τ is considered a parameter, then (7) represents a pencil of cubics on the nine inflexions. The degenerate curves of the pencil are the inflexional triangles and they intersect the proper curves at the inflexions of the proper curves.

8. An irrationality of the type discussed in section I occurs if one of the involutory† collineations in G_{18} is considered. One such collineation is $x_0(-u) = x_0(u), x_1(-u) = x_2(u), x_2(-u) = x_1(u)$, where the center is the inflexion $(0, 1, -1)$ and the axis upon which Q_3 is projected is the harmonic polar $x_1 - x_2 = 0$. A pencil of lines

$$t_1 x_0 + t_2(x_1 + x_2) = 0 \tag{8}$$

on the center establishes a two-to-one correspondence between the points of Q_3 and the points on the axis. The irrationality separating two corresponding points of Q_3 is from (7) and (8) found to be

$$\frac{x_1}{x_2} = \frac{t^3 - 3\tau t^2 + 2 \pm \sqrt{-3t^2(t^4 + 2\tau t^3 - 3\tau^2 t^2 - 4t + 4\tau)}}{2(t^3 - 1)} \quad (t = t_1/t_2).$$

Let

$$t^4 + 2\tau t^3 - 3\tau^2 t^2 - 4t + 4\tau = (t - \tau)(t^3 + 3\tau t^2 - 4) \equiv a_t^4.$$

The values of t for which $a_t^4 = 0$ give the double points of the involution, which are the points of contact of the tangents drawn from the inflexion. Since one of these tangents is the flex tangent which has the parameter $t = \tau$, a_t^4 can be written in the factored form given. The four flex lines on the point $(0, 1, -1)$ have parameters given by

$$\alpha_t^4 \equiv t_2(t_1^3 - t_2^3) = 0, \ddagger$$

an equianharmonic quartic. If α_t^4 is polarized with respect to τ, it becomes evident that $a_t^4 = (t\tau)\alpha_\tau \alpha_t^3$. Hence the irrationality $\sqrt{a_t^4}$ is of exactly the type given in section I. A geometric interpretation of the result is that *at an inflexion I of Q_3, the three tangents from I other than the flex tangent are given by the first polar of the flex tangent at I as to the four flex lines on I.*

* B., section 1. By using $\tau = -2\tau$ in (7) Bianchi's form is obtained.

† Segre: *Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes*, Mathematische Annalen, vol. 27, 1886.

‡ The inflexions of Q_3 can be permuted in 216 different ways. If one is fixed as in the present case, the four flex lines on it can at the most be permuted among themselves. It is therefore to be expected that their parameters form an equianharmonic quartic invariant under a tetrahedral group.

For the irrationality $\sqrt{a_t^4}$

$$g_2 = \frac{3\tau}{4}(\tau^3 + 8), \qquad g_3 = \frac{1}{8}(\tau^6 - 20\tau^3 - 8), \qquad \Delta = -27(1 - \tau^3)^3.$$

The syzygy (4) becomes

$$J : J - 1 : 1 = \tau^3(\tau^3 + 8)^3 : (\tau^6 - 20\tau^3 - 8)^2 : 64(\tau^3 - 1)^3,$$

the form of the modular equation* given by Klein. Hence τ is the tetrahedral irrationality.†

III. The elliptic norm curve of fourth order, Q_4, in three-dimensional space, S_3

9.‡ The coördinates of a point on the elliptic norm curve of fourth order, Q_4, in three-dimensional space S_3 are given by

$$x_a(u) = c_a \prod_{\mu=0}^{3} \sigma_{a/4+1/2,\,(\mu+1/2)/4}(u \mid \omega_1, \omega_2) \qquad (\alpha = 0, \cdots 3),$$

where c_a is an undetermined factor and

$$\sigma_{a/4,\,\mu/4}(u \mid \omega_1, \omega_2) \equiv e^{[(\lambda\eta_1+\mu\eta_2)/4][u-(\lambda\omega_1+\mu\omega_2)/8]}\,\sigma\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{4} \,\middle|\, \omega_1, \omega_2\right).$$

To obtain a convenient expression for c_a, Klein introduces the transformed periods $\bar{\omega}_1$, $\bar{\omega}_2$, such that $\bar{\omega}_1 = \omega_1$, $\bar{\omega}_2 = \omega_2/4$. Because of the Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$, the corresponding substitution for η is $\bar{\eta}_1 = 4\eta_1$, $\bar{\eta}_2 = \eta_2$. The coördinates are finally given in the form

$$x_a(u \mid \omega_1, \omega_2) = a_a\, e^{-[(\eta_1-4\eta_1)/2\omega_1]u^2+([(a+2)/4]\bar{\eta}_1+\bar{\eta}\ /2)u}$$
$$\cdot\, e^{-([(a+2)/4]\bar{\eta}_1+\bar{\eta}_2/2)([(a+2)/8]\omega_1+\omega_2/16)}$$
$$\cdot\, \sigma\left(u - \frac{\alpha+2}{4}\omega_1 - \frac{\mu+\frac{1}{2}}{4}\omega_2 \,\middle|\, \omega_1, \frac{\omega_2}{4}\right),$$

where

$$a_a = e^{-1/2[(\pi i a)/4+(5\pi i)/2]}\mathrm{X},$$

X being a function of the periods, independent of u and α. The notation indicates that $x_a(u)$ depends upon the periods ω_1, ω_2, the σ functions upon the periods ω_1, $\omega_2/4$. In the coördinate system used $x_{a+4}(u) = x_a(u)$.

Four points u_1, u_2, u_3, u_4, of Q_4 are on a plane if

$$(9) \qquad u_1 + u_2 + u_3 + u_4 \equiv 0 \qquad (\mathrm{mod}\ \omega_1, \omega_2).$$

* K. F., I, p. 104.

† See B., section 1, for further discussion of τ.

‡ §§ 9 and 10 do little more than give for the case $n = 4$ the general theory stated by Klein for Q_n (K. F., II, pp. 261–265).

There are 16 singular points (points of hyperosculation), which lie by fours on the faces of a tetrahedron T. These sets of fours are

$$\text{(10)} \qquad \frac{1}{2}P, \quad \frac{\omega_1}{4}+\frac{1}{2}P, \quad \frac{\omega_2}{4}+\frac{1}{2}P, \quad \frac{\omega_1+\omega_2}{4}+\frac{1}{2}P,$$

where

$$\frac{1}{2}P = 0, \quad \frac{\omega_1}{2}, \quad \frac{\omega_2}{2}, \quad \frac{\omega_1+\omega_2}{2}.$$

10. The curve Q_4 admits a collineation group G_{32} determined by the 32 substitutions

$$\text{(11)} \qquad u' = \pm u + \frac{p\omega_1 + q\omega_2}{4}{}^* \qquad (p, q = 0, 1, 2, 3).$$

Under G_{32}, Q_4 is invariant; and under (11), (9) is invariant. The G_{32} is defined abstractly by K_1, K_2, K_3, such that

$$K_1^2 = K_2^4 = K_3^4 = 1, \qquad K_2 K_3 = K_3 K_2,$$
$$K_1 K_2 = K_2^3 K_1, \qquad K_1 K_3 = K_3^3 K_1.$$

These generators can be taken to be

$$\text{(12)} \qquad u' = -u, \qquad u' = u - \frac{\omega_1}{4}, \qquad u' = u - \frac{\omega_2}{4}, \qquad \text{i. e.,}$$

$$x_\alpha(-u) = x_{4-\alpha}(u), \qquad x_\alpha\left(u - \frac{\omega_1}{4}\right) = x_{\alpha+1}(u),$$
$$x_\alpha\left(u - \frac{\omega_2}{4}\right) = \epsilon^\alpha x_\alpha(u),\dagger \qquad (\epsilon = e^{(2\pi i)/4}).$$

The G_{32} contains an invariant Abelian G_{16} generated by K_2 and K_3. This G_{16} contains three pairs of cyclic G_4's, each pair having a common element of period two. The three elements of period two, K_2^2, K_3^2, $K_2^2 K_3^2$, and identity constitute a four-group whose parametric expression is

$$u' = u + \tfrac{1}{2}P.$$

By adjoining K_1 four new elements of period two, which with the four-group form an Abelian G_8, are obtained. Their parametric expression is

$$u' = -u + \tfrac{1}{2}P.$$

From (10) they are evidently reflexions in vertex and opposite plane of T. Thus the unique tetrahedron T can be determined from the K's. By the transformations

$$y_0 = x_0 + x_2, \qquad y_1 = x_1 + x_3, \qquad y_2 = x_0 - x_2, \qquad y_3 = x_1 - x_3,$$

* K. F., II, pp. 237–242; and Segre, loc. cit.

† For factors of proportionality see K. F., II, p. 264.

T is made the reference tetrahedron and K_1, K_2, K_3 take the forms

$$\begin{array}{ll} C_1 & C_2 \\ y_0(-u) = y_0(u) & y_0\left(u - \frac{\omega_1}{4}\right) = y_1(u) \\ y_1(-u) = y_1(u) & y_1\left(u - \frac{\omega_1}{4}\right) = y_0(u) \\ y_2(-u) = y_2(u) & y_2\left(u - \frac{\omega_1}{4}\right) = y_3(u) \\ y_3(-u) = -y_3(u) & y_3\left(u - \frac{\omega_1}{4}\right) = -y_2(u) \end{array}$$

$$\begin{array}{c} C_3 \\ y_0\left(u - \frac{\omega_2}{4}\right) = y_2(u) \\ y_1\left(u - \frac{\omega_2}{4}\right) = iy_3(u) \\ y_2\left(u - \frac{\omega_2}{4}\right) = y_0(u) \\ y_3\left(u - \frac{\omega_2}{4}\right) = -iy_1(u). \end{array}$$

11. The pencil of quadrics on Q_4 has a unique self-polar tetrahedron whose vertices are the vertices of cones on Q_4. Since the lines joining corresponding points u, $-u$, and v, $-v$ under $u' = -u$ intersect, this tetrahedron is evidently T. Thus G_{32} leaves the pencil of quadrics unaltered and permutes the four cones under a four-group, which is the factor group of G_{32} with reference to the G_8 mentioned in § 10.

If the cone with vertex opposite $y_3 = 0$ is

(13) $$ay_0^2 + by_1^2 + cy_2^2 = 0,$$

the others furnished by the four-group are

(14) $$by_0^2 + ay_1^2 + cy_3^2 = 0,$$

(15) $$cy_0^2 + ay_2^2 - by_3^2 = 0,$$

(16) $$cy_1^2 - by_2^2 + ay_3^2 = 0.$$

These are in a pencil if

(17) $$a^2 = b^2 + c^2$$

or if

(18) $$a = \tau_1^2 + \tau_2^2, \qquad b = \tau_1^2 - \tau_2^2, \qquad c = 2\tau_1 \tau_2.$$

As τ varies the family of Q_4's, each of which admits the same G_{32}, is obtained. This family lies on the surface

(19) $$y_0^4 - y_1^4 - y_2^4 + y_3^4 = 0,$$

obtained by eliminating τ.

12. Let $P_0 \cdots P_3$ be the vertices of T; $L_i = \overline{P_0 P_i}$, $L_i' = \overline{P_j P_k}$, $(i, j, k = 1, 2, 3)$, pairs of opposite edges of T. Let a cyclic G_4 (§ 10) be generated by C_2 or $u' = u - \omega_1/4$. Then C_2 is not perspective, for if it had a pencil of fixed planes, the pencil would cut the curve in sets of four conjugate co-planar points, whereas the general set u, $u - \omega_1/4$, $u - 2\omega_1/4$, $u - 3\omega_1/4$ is not coplanar. Hence C_2 has a fixed tetrahedron T_i whose vertices are on a line pair L_i, L_i' of T, since C_2^2 is a reflexion in such a line pair. The reflexions in the other line pairs leave the cyclic G_4 unaltered, whence the vertices of T_i on L_i and L_i' are harmonic with those of T. The second cyclic G_4 containing C_2^2 (§ 10) has a fixed tetrahedron T_i'. Thus the vertices of T, T_i, T_i' on L_i, L_i' form three mutually harmonic pairs on each line L_i. Since the cyclic G_4's are invariant under G_{32}, the six tetrahedra T_i, T_i' are invariant under G_{32}. The edges of the six tetrahedra T_i, T_i', exclusive of the edges common to T, form six skew quadrilaterals.

The elements of G_{32} not in G_{16} are involutory. The four of type

$$u' = -u + \tfrac{1}{2}P$$

are reflexions in vertex and opposite plane of T (vide § 10). The remaining twelve are $u' = -u + \frac{1}{4}P$ $(\frac{1}{4}P \mp \frac{1}{2}P)$. Since two pairs of corresponding points of Q_4 are not in general coplanar, these twelve collineations are reflexions in skew line pairs. If $u' = -u + \frac{1}{4}P$ is a reflexion in the line pair M_i, M_i', then, since it transforms $u' = -u + \frac{1}{2}P'$ into $u' = -u + \frac{1}{2}P + \frac{1}{2}P'$, it must interchange the vertices of T, i. e., M_i, M_i' lie across two skew edges of T say L_i and L_i'. The product of $u' = -u + \frac{1}{4}P$ and $u' = u + \frac{1}{2}P$, i. e., $u' = -u + \frac{3}{4}P$ will be another of the twelve involutions whose line pair N_i, N_i' will cross both L_i, L_i' and M_i, M_i'. Thus the twelve involutions determine six pairs of line pairs which form six skew quadrilaterals $\overline{T}_i$ with vertices on L_i, L_i'. These six quadrilaterals separate into three pairs $\overline{T}_i$, $\overline{T}_i'$, the two of a pair having their vertices on the same line pair L_i, L_i'.

The involution on M_i, M_i' leaves unaltered the line L_i and the tetrahedra T_i, T_i'. Since it cannot interchange all three pairs of vertices of T_i, T_i', T on L_i the pair must cross L_i at the vertices of T_i or T_i'. Hence: *The six skew quadrilaterals T_i, T_i' and the six skew quadrilaterals $\overline{T}_i$, $\overline{T}_i'$ have the same vertices.*

If a_i, b_i are the pairs of vertices on L_i, a_i', b_i' the pairs on L_i', and if the quadrilaterals T_i, T'_i are formed by joining a_i to a'_i, b_i to b'_i, then the quadrilaterals $\overline{T}_i$, $\overline{T}'_i$ are formed by joining a_i to b'_i, b_i to a'_i.

13. If an elliptic quartic degenerates, at least two of the four cones on it must coincide. From equations (13)–(16) it can be seen that this requires a or b or c to be zero. Then two pairs of cones each coincide into pairs of planes and the quartic is a skew quadrilateral. There are six such Q_4's in the family, corresponding to the values of the modulus given by

$$\alpha_\tau^6 \equiv \tau_1^5 \tau_2 - \tau_1 \tau_2^5 = 0. \tag{20}$$

On the other hand, the six quadrilaterals T_i admit G_{32} (§ 12) and occur in the family. Therefore the quadrilaterals T_i are on the quartic surface (19).

The parametric representation shows that the involution on M_i, M'_i has four fixed points on each Q_4. This line pair meets each T_i in four points. It has been shown above that for T_i and T'_i these four points are two vertices or double points. For T_j, T'_j the four points are paired on opposite edges. Thus M_i has more than four points in common with the surface (19) and therefore lies on it. But the surface has just 48 lines determined by the separation of its terms into two sets of two. Hence *the twelve skew quadrilaterals T_i, T'_i; $\overline{T}_i$, $\overline{T}'_i$ are the* 48 *lines of the quartic locus of the family of Q_4's which admit the given G_{32}. The six quadrilaterals T_i, T'_i are the degenerate curves of the family.**

14. In order to introduce the irrationality discussed in section I, the family of Q_4's will be projected into a family of doubly-covered rational curves as was done in the case of Q_3 (section II). In that simple case the projected family lay on a single line. The projection was made by means of one of the involutions which belonged to every curve. In the present case there are two types of involution in G_{32} which are not in G_{16}. By using the type determined by a vertex and opposite plane of T, the family of Q_4's is projected upon a family of doubly-covered conics. By using the type determined by two skew lines of the quadrilateral $\overline{T}_i$ the projection is upon a family of doubly covered lines all of which lie on one line of the pair.

For the first type, if $y_3 = 0$ is the plane of the projection, the family of doubly-covered conics is from (13)

$$ay_0^2 + by_1^2 + cy_2^2 = 0. \tag{21}$$

In order to introduce a parameter t on the conic, which is covariantly related to the modulus τ of the quartic, it is noted that the reference triangle cuts out on the conic an octahedral sextic which can be identified with α_τ^6, the one which determines the degenerate curves. Then the conic can be written as follows:

* Segre, loc. cit., and *Concerning the twisted biquadratic*, by J. C. Kluyver, American Journal of Mathematics, vol. 19 (1897), p. 319 ff.

$$
(22)\qquad
\begin{aligned}
y_0 &= \sqrt{\tau_1^2 - \tau_2^2}\,\sqrt{2\tau_1\tau_2}\,(t_1^2 + t_2^2),\\
y_1 &= i\,\sqrt{2\tau_1\tau_2}\,\sqrt{\tau_1^2 + \tau_2^2}\,(t_1^2 - t_2^2),\\
y_2 &= i\,\sqrt{\tau_1^2 + \tau_2^2}\,\sqrt{\tau_1^2 - \tau_2^2}\,(2t_1 t_2),
\end{aligned}
$$

and from (14)

$$y_3 = \pm 2\,\sqrt{2\,(t\tau)\,\alpha_\tau^3\,\alpha_\tau^3}.$$

The geometric interpretation of the radical in t can be stated thus: *If an elliptic quartic lies on a cone, the six generators on pairs of points of hyperosculation form an octahedral sextic, and the four generators on single points of hyperosculation are such that any three form the cubic polar of the remaining one as to the sextic.*

The expression for $\tau = \tau_1/\tau_2$ as an elliptic modular function can be obtained from (21) and (19) in the form

$$\tau = \frac{\tau_1}{\tau_2} = \frac{-\,y_2^2 + \sqrt{y_2^4 - y_0^4 + y_1^4}}{y_0^2 + y_1^2} = \frac{-\,y_2^2 + y_3^2}{y_0^2 + y_1^2}.$$

If this is evaluated for $u = 0$ and expressed in terms of x_i (vide § 10), the following expression is obtained

$$\tau = \frac{-\,[x_0(0) - x_2(0)]^2 + [x_1(0) - x_3(0)]^2}{[x_0(0) + x_2(0)]^2 + [x_1(0) + x_3(0)]^2}.$$

The modular equation satisfied by τ can be obtained by forming g_2, g_3, and Δ for the irrationality $\sqrt{2\,(t\tau)\,\alpha_\tau^3\,\alpha_t^3}$ as in section I by letting

$$n = 6, \qquad \alpha_t^6 = t_1^5\,t_2 - t_1\,t_2^5.$$

The equation appears in the well-known octahedral form.*

The presence of the radicals $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ in the parametric form (22) of Q_4 is due to the fact that in the plane $y_3 = 0$ there are four points of hyperosculation determined by $t = \pm\,\tau,\ \pm\,1/\tau$, the roots of $(t\tau)\,\alpha_\tau^3\,\alpha_t^3 = 0$. One of these, $t = \tau$, has been isolated by the choice of the parameter t.

A striking form of the elliptic integral can be deduced from the type

$$u = \int \frac{(t\,dt)}{\sqrt{(t\tau)\,\alpha_\tau^3\,\alpha_t^3}}.$$

The family of Q_4's has been projected into the family of conics (21) which are intimately related to the quartic section of the family of Q_4's, made by the plane of projection. This quartic section† is

* K. F., I, p. 20.

† *Über Untersuchung und Aufstellung von Gruppe und Irrationalität regulärer Riemann'schen Flächen* and *Notiz über eine reguläre Riemann'sche Fläche vom Geschlechte drei und die zugehörige "Normalcurve" vierter Ordnung,* by W. Dyck in Mathematische Annalen, vol. 17 (1880), pp. 473 and 510. The connection is here made with the elliptic modular function, but the system of conics mentioned above is not considered. The normal forms of the elliptic integral are to be contrasted with those described in an article by Klein in the same volume.

(23) $$a_y^4 \equiv y_0^4 - y_1^4 - y_2^4 = 0$$

or parametrically

(24) $$y_0 = \sqrt{a}, \qquad y_1 = i\sqrt{b}, \qquad y_2 = i\sqrt{c}.$$

The family (21) is evidently the system of polar conics as to a_y^4, of a point y on $a_y^4 = 0$. Moreover $a_y^4 = 0$ is the envelope of this quadratic family of conics (21), which appears therefore as an isolated Steiner system of the quartic curve.

Consider the integral

(24) $$I = \int \frac{(y\,x\,xd)}{\sqrt{a_y^3\,a_x \cdot (aa'\,a'')\,a_y^3\,(a'\,a''\,yx)^3}},$$

where x and y are subject to the relations $a_y^4 = 0$, $a_y^2\,a_x^2 = 0$. The integral I depends first upon the choice of the point y on $a_y^4 = 0$, secondly upon $x_0 : x_1 : x_2$, a point which is required to be on (21), $a_y^2\,a_x^2 = 0$. That I is an elliptic integral follows from the fact that if a parameter is introduced on the rational curve the radical reduces to one of degree four. This can easily be verified. Let

$$a_1 = t_1^2 + t_2^2, \qquad b_1 = t_1^2 - t_2^2, \qquad c_1 = 2t_1\,t_2.$$

Then with y expressed as in (24) and $a_y^2\,a_x^2 = 0$ as in (22) it is found that

$$(y\,x\,dx) = -4\sqrt{a}\,\sqrt{b}\,\sqrt{c}\,(t\tau)^2\,(t\,dt).$$

Since

$$(a'\,a''\,u)^4 = 2\,(u_0^4 - u_1^4 - u_2^4),$$

$$a_y^3\,a_x = 4\,(y_0^3\,x_0 - y_1^3\,x_1 - y_2^3\,x_2),$$

(25) $$(aa'\,a'')\,a_y^3\,(a'\,a''\,u)^3 = 32\,(y_0^3\,u_0^3 + y_1^3\,u_1^3 + y_2^3\,u_2^3).$$

Instead of the line coördinates introduce the values

$$u_0 = y_1\,x_2 - y_2\,x_1 = \sqrt{a}\,(-bc_1 + b_1\,c) = \sqrt{a}\,(t\tau)\,(t_1\,\tau_1 + t_2\,\tau_2), \quad \text{etc.}$$

Then (25) becomes

(25′) $$(aa'\,a'')\,a_y^3\,(a'\,a''\,u)^3 = \frac{16}{5}\,(t\tau)^3\,\alpha_\tau^3\,\alpha_t^3,$$

and

$$a_y^3\,a_x = 8\sqrt{a}\,\sqrt{b}\,\sqrt{c}\,(t\tau)^2.$$

The integral I can now be written

$$I = -\frac{\sqrt{5}}{\sqrt[4]{abc}} \int \frac{(t\,dt)}{2\sqrt{2\,(t\tau)\,\alpha_\tau^3\,\alpha_t^3}}.$$

If $a_y^4 = 0$ is the Dyck quartic automorphic under a collineation group G_{96}, the integral

$$I = \int \frac{(y\,x\,dx)}{\sqrt{a_y^3\,a_x \cdot (aa'\,a'')\,a_y^3\,(a'\,a''\,yx)^3}}$$

is an elliptic integral of the first kind with moduli $y_0 : y_1 : y_2$, *and variables* $x_0 : x_1 : x_2$ *on the polar conic of* y *as to the quartic. This integral is unaltered if the variables and moduli are cogrediently transformed under* G_{96}.

If $a_y^4 = 0$ *is the Klein quartic invariant under the collineation group* G_{168} *the integral* I *has the same properties and covariant character.*

The last statement is given here without proof. The theorem indicates the possibility of extending the canonical form obtained for genus zero to larger values of the genus and gives the form applicable to genus three.

For the second type of projection mentioned in the beginning of this paragraph, the involution $C_1\,C_2$ or parametrically $u' = -u + \omega_1/4$ is used. A pencil of planes on one fixed line of $C_1\,C_2$ is given by

$$y_0 + y_1 + i\sqrt{\tau}\lambda\,(y_2 + y_3) = 0. \tag{26}$$

By eliminating y_0/y_3, y_1/y_3 from (26), (13), and (14) the value of y_2/y_3 for the points of intersection of the plane (26) and Q_4 is found to be

$$\frac{y_2}{y_3} = \frac{[a+b]\,[c^2 - \tau^2\lambda^4(a-b)^2] \mp 2\sqrt{\tau}\lambda c\,\sqrt{8\,(\lambda\tau)\,\alpha_\tau^3\,\alpha_\lambda^3}}{\tau^2\lambda^4 c^2(a-b) - 2\tau\lambda^2 c\,(a-b)^2 + c^2(a+b)}. \tag{27}$$

The irrationality in (27) which separates corresponding points of Q_4 is again in the canonical form.*

15. Since τ is an elliptic modular function and the octahedral irrationality, it admits the group Γ_{24} the principal subgroup of Γ_{48}, which is generated by

$$\mathbf{S}\quad \begin{aligned} \omega_1' &= \omega_1 + \omega_2 \\ \omega_2' &= \omega_2 \end{aligned} \qquad \mathbf{T}\quad \begin{aligned} \omega_1' &= -\,\omega_2 \\ \omega_2' &= \omega_1, \end{aligned}$$

where the determinants of the transformations are congruent to identity (mod 4). The effect of **S** and **T** on the coördinates of a point of Q_4 is given by

$$\mathbf{S}_1 : y_0' = \sqrt{i}y_2, \quad y_1' = iy_1, \quad y_2' = \sqrt{i}y_0, \quad y_3' = iy_3$$

$$\mathbf{T}_1 : y_0' = y_0, \quad y_1' = y_2, \quad y_2' = y_1, \quad y_3' = -\,iy_3;\dagger$$

on the modulus τ, by

* The explicit equations of the quadrics generated by the lines joining corresponding points of Q_0 under $C_1\,C_2$ and the eleven other involutions of the same type can be derived (vide also Segre, loc. cit.). The parameters of these six quadrics as members of the pencil (13) + μ (14) = 0 form an octahedral sextic.

† K. F., II, p. 299, (12) and (13). There is a misprint in (12) but the form is given correctly on p. 297. Klein gives the transformation on x's for Q_n.

$$S_2 : \tau' = \frac{\tau - i}{\tau + i} \qquad \text{or} \qquad \tau' = \frac{\tau + i}{\tau - i},$$

$$T_2 : \tau' = \frac{\tau + 1}{\tau - 1} \qquad \text{or} \qquad \tau' = \frac{\tau - 1}{\tau + 1}.$$

The first substitutions of S_2 and T_2 are sufficient to generate the octahedral group G_{24} on τ.

By the addition of G_{24} to G_{32} a $G_{32 \cdot 24}$ is obtained under which the quartics of the family are at the most permuted among themselves.

If now the surface

$$(19) \qquad y_0^4 - y_1^4 - y_2^4 + y_3^4 = 0$$

is considered, it will readily be seen that (19) is unaltered by the twenty-four permutations and the four letters y_i combined with the roots of -1, of which there are three choices. Hence a $G_{24 \cdot 4 \cdot 4 \cdot 4}$ leaves (19) unaltered. But a $G_{32 \cdot 24}$ has already been accounted for. The one type of transformation not yet included is

$$C_4 \qquad y_0' = y_0, \qquad y_1' = y_1, \qquad y_2' = y_2, \qquad y_3' = iy_3.$$

The surface (19) is invariant under C_4 but its two sets of twenty-four lines are interchanged, i. e., the axes of the twelve involutory collineations of type $C_1 C_2$ which form the skew quadrilaterals $\overline{T}_i$, $\overline{T}'_i$, are transformed into the degenerate curves forming the skew quadrilaterals T_i, T'_i and vice versa.* This indicates the existence of two families of Q_4's on the surface (19) arranged as are the generators of a quadric in that through every point of the surface passes a curve of each family. But a curve of one family cuts every curve of the other family in eight points.

IV. The elliptic norm curve of the fifth order, Q_5, in four-dimensional space, S_4

16. The reader is referred to the frequently cited article of Bianchi for the preliminary work needed in the discussion of the elliptic curve of fifth order, Q_5, in four-dimensional space S_4. In accordance with the notation of that article, the six fundamental pentahedra will be designated by P_∞, P_i ($i = 0, \cdots, 4$). The collineations determined by the substitutions $u' = -u$, $u' = u - \omega_1/5$, $u' = u + \omega_2/5$ will be represented by C_1, C_2, C_3 respectively. These are the generators of the fifty collineations, G_{50}, under which Q_5 is invariant. The five quadrics whose common intersection is Q_5 are

$$\phi_i \qquad \tau_1 \tau_2 x_i^2 + \tau_1^2 x_{i+2} x_{i+3} - \tau_2^2 x_{i+1} x_{i+4} = 0\dagger \quad (i = 0, \cdots 4;\ x_{i+5} = x_i).$$

* See also Kluyver, loc. cit.

† B., § 13. The a of Bianchi is replaced by τ_1 / τ_2.

In order to develop the elliptic irrationality it is convenient to introduce coördinates suggested by the involution C_1. Let

$$y_0 = x_0, \qquad y_1 = x_1 + x_4, \qquad y_2 = x_2 + x_3,$$
$$z_1 = x_1 + x_4, \qquad z_2 = -x_2 + x_3.^*$$

Then C_1 has a plane π of fixed points determined by $z_1 = z_2 = 0$, in which the coördinates are $y_0 : y_1 : y_2$; and a skew line L of fixed points determined by $y_0 = y_1 = y_2 = 0$, in which the coördinates are $z_1 : z_2$. The explicit form of C_1 in the new coördinate system† is

$$C_1': \qquad y_i(-u) = -y_i(u), \qquad z_j(-u) = z_j(u) \quad (i = 0, 1, 2; j = 1, 2).$$

Since corresponding points in the involution are cut out by a pencil of S_3's on π, such points will have for their coördinates the same y_i's and the z_j's differing by the sign of a radical. The radical can easily be obtained.

17. The quadrics ϕ_i can be rewritten in terms of y and z.

$$\phi_0'. \qquad 4\tau_1\tau_2 y_0^2 + \tau_1^2 y_2^2 - \tau_2^2 y_1^2 = (\tau_1 z_2 + \tau_2 z_1)(\tau_1 z_2 - \tau_2 z_1),$$

$$\phi_1'. \qquad \tau_1\tau_2 y_1^2 + \tau_1^2 y_1 y_2 - 2\tau_2^2 y_0 y_2 = \tau_1 z_1(\tau_1 z_2 - \tau_2 z_1),$$

$$\phi_2'. \qquad \tau_1\tau_2 y_2^2 + 2\tau_1^2 y_0 y_1 - \tau_2^2 y_1 y_2 = -\tau_2 z_2(\tau_1 z_2 - \tau_2 z_1),$$

$$\phi_3'. \qquad 2\tau_2^2 z_2 y_0 + (\tau_1^2 z_2 + 2\tau_1\tau_2 z_1) y_1 - \tau_1^2 z_1 y_2 = 0,$$

$$\phi_4'. \qquad 2\tau_1^2 z_1 y_0 + \tau_2^2 z_2 y_1 + (\tau_2^2 z_1 + 2\tau_1\tau_2 z_2) y_2 = 0.$$

Since ϕ_3' and ϕ_4' are homogeneous in y_i, they can be solved for y_i.

$$y_0 = -\rho\tau_1\tau_2[\tau_2^2 z_1^2 + 3\tau_1\tau_2 z_1 z_2 + \tau_1^2 z_2^2],$$
$$y_1 = \rho[\tau_1^4 z_1^2 + \tau_2^4 z_1 z_2 + 2\tau_1\tau_2^3 z_2^2],$$
$$y_2 = \rho[2\tau_1^3\tau_2 z_1^2 + \tau_1^4 z_1 z_2 - \tau_2^4 z_2^2],$$

where ρ is an undetermined factor of proportionality. By substituting these values in ϕ_0', ϕ_1', ϕ_2' the condition which ρ must satisfy is found to be

$$\rho^2\{z_1^3[4\tau_1^3\tau_2^6 + 3\tau_1^8\tau_2] + z_1^2 z_2[18\tau_1^4\tau_2^5 + \tau_1^9] + z_1 z_2^2[18\tau_1^5\tau_2^4 - \tau_2^9] + z_2^3[4\tau_1^6\tau_2^3 - 3\tau_1\tau_2^8]\} = \tau_1 z_2 - \tau_2 z_1. \tag{28}$$

Let $\alpha_z^{12} \equiv z_1 z_2[z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}]$. Then (28) can be written

$$4\rho^2\alpha_\tau^9\alpha_z^3 = (\tau z). \tag{28'}$$

$$\therefore \quad \rho = \frac{(\tau z)}{2\sqrt{(\tau z)\alpha_\tau^9\alpha_z^3}}.$$

---

* Whenever a collineation is indicated by an accented letter its form in terms of y and z is meant.

† K. F., II, p. 267 especially (13).

If this value of ρ is used, Q_5 can be written parametrically in terms of z.

$$
(29)\qquad
\begin{aligned}
y_0 &= -\tau_1\tau_2(\tau z)[\tau_2^2 z_1^2 + 3\tau_1\tau_2 z_1 z_2 + \tau_1^2 z_2^2],\\
y_1 &= (\tau z)[\tau_1^4 z_1^2 + \tau_2^4 z_1 z_2 + 2\tau_1\tau_2^3 z_2^2],\\
y_2 &= (\tau z)[2\tau_1^3\tau_2 z_1^2 + \tau_1^4 z_1 z_2 - \tau_2^4 z_2^2],\\
z_1 &= 2z_1\sqrt{(\tau z)\alpha_\tau^9\alpha_z^3},\\
z_2 &= 2z_2\sqrt{(\tau z)\alpha_\tau^9\alpha_z^3}.
\end{aligned}
$$

Now consider a pencil of S_3's on π, given by

$$
(30)\qquad t_2 z_1 = t_1 z_2.
$$

If in (29) z is replaced by t, (29) gives the coördinates of pairs of points in the involution, which by (30) project into one point on L. The notation is then exactly analogous to that used in earlier sections of this paper. But z can equally well be thought of as the variable parameter along a line. In either notation the irrationality which appears is of the type discussed in section I. The icosahedral equation can be developed in terms of τ, and τ is therefore the icosahedral irrationality.*

18. Now project Q_5 by planes on L, upon π. Since a plane in S_4 in general meets another plane in S_4 in only one point, involutory pairs on Q_5 project into single points. The locus of these points on π is the conic whose parametric expression (parameter z) is furnished by y_i of (29). The point equation of this doubly-covered conic is

$$
(31)\qquad 2\tau_1^2\tau_2^2 y_0^2 - \tau_1\tau_2^3 y_1^2 + \tau_1^3\tau_2 y_2^2 + \tau_1^4 y_0 y_1 - \tau_1^2\tau_2^2 y_1 y_2 + \tau_2^4 y_0 y_2 = 0.
$$

The line L meets Q_5 in a singular point of Q_5, whose elliptic parameter is $u = 0$, and whose parameter in (29) is $z = \tau$. The locus of the projection of this point on π is the rational sextic

$$
(32)\qquad
\begin{aligned}
y_0 &= -5z_1^3 z_2^3,\\
y_1 &= z_1^6 + 3z_1 z_2^5,\\
y_2 &= 3z_1^5 z_2 - z_2^6.
\end{aligned}
$$

The conic (31) is the osculant conic of (32) at the point $z = \tau$. With every Q_5 in the family is associated a point of the sextic (32). Hence

The family of elliptic quintics in S_4, which admit a given G_{50} of collineations, is projected from the fixed line upon the fixed plane of one of the twenty-five involutory collineations of G_{50} into a family of osculant conics of a rational

* See B. also.

sextic in that plane. Through every point of the plane pass four conics of the system.

19. A consideration of the projection of Q_5 leads to a determination of the order r of the surface ψ which is the locus of the family of Q_5's. Since L is on ψ, an S_3 on L cuts ψ in L and some curve C of order $r-1$, which meets L in s points. In a given S_3, an S_2 on L cuts C in s points on L and $r-1-s$ further points. But it has been noted that through every point of π pass four conics and that to every point of a conic corresponds two points of a particular Q_5. Hence to every point of the plane corresponds eight points of the surface ψ. Therefore

$$r - 1 - s = 8.$$

Now the S_3 on L, cutting ψ in L and C contains two directions on the surface ψ at every point of intersection of L and C, and therefore contains the tangent S_2 at such a point. But the section of S_3 by π is a line and the projection of points of contact of tangent planes is the sextic (32). Since a line cuts a sextic in six points, $s = 6$.

$$\therefore \quad r = 15.$$

20. In the preceding paragraph one line, L, on ψ has been mentioned. This however is but one of twenty-five such lines which are the fixed lines of the involutory collineations in G_{50}. These lines lie by fives on S_3's. Five S_3's containing such an arrangement of these lines form one of the six fundamental pentahedra.

Another set of lines on ψ are those composing the degenerate Q_5's obtained by letting τ equal the roots of

$$\alpha_\tau^{12} \equiv \tau_1 \tau_2 (\tau_1^{10} + 11\tau_1 \tau_2 - \tau_2^{10}) = 0.$$

Each degenerate curve is a skew pentagon. The twelve pentagons can be arranged in six pairs, each pair forming edges of a fundamental pentahedron. The pairs are given by

$$\tau_1 \tau_2 = 0$$

and

$$[\tau_1 - \epsilon^\nu (\epsilon + \epsilon^4) \tau_2] [\tau_1 - \epsilon^\nu (\epsilon^2 + \epsilon^3) \tau_2] = 0$$
$$(\nu = 0, \cdots 4; \ \epsilon = e^{(2\pi i)/5}).$$

21. In a study of the configurations in the plane π, the transformations arising through the modular substitutions $\boldsymbol{S}$ and $\boldsymbol{T}$ are valuable. Bianchi has given the two transformations also called $\boldsymbol{S}$ and $\boldsymbol{T}$ in terms of x. They with C_1, C_2, C_3 generate $G_{50 \cdot 60}$ under which the members of the family of Q_5's in S_4 are at the most permuted. Under $G_{50 \cdot 60}$, G_{50} is an invariant subgroup. The factor group is a G_{60} isomorphic with the modular substitu tions. The subgroup of $G_{50 \cdot \cdot 60}$ leaving C_1 unaltered is a $G_{2 \cdot 60}$. It has an invariant G_2, $\{1, C_1\}$, whose factor group is isomorphic with the G_{60} already mentioned. This may be represented either by the transformations on $z_1 : z_2$ or those on $y_0 : y_1 : y_2$. The substitutions in π which generate such a G_{60} are obtained by setting $z_1 = z_2 = 0$ in T and $(C_1 S^2)^3 \equiv S_1'$. The explicit expressions are

	S_1'	T'
y_0'	$= y_0$	$= y_0 + y_1 + y_2$
y_1'	$= \epsilon^2 y_1$	$= 2y_0 + (\epsilon + \epsilon^4) y_1 + (\epsilon^2 + \epsilon^3) y_2$
y_2'	$= \epsilon^3 y_2$	$= 2y_0 + (\epsilon^2 + \epsilon^3) y_1 + (\epsilon^4 + \epsilon) y_2$
z_1'	$= -\epsilon^2 z_1$	$= (\epsilon - \epsilon^4) z_1 + (\epsilon^3 - \epsilon^2) z_2$
z_2'	$= -\epsilon^3 z_2$	$= (\epsilon^3 - \epsilon^2) z_1 + (\epsilon^4 - \epsilon) z_2$*

Let us now consider the degenerate Q_5's and their projections in π. To particularize let $\tau_1 = 0$ or $\tau_2 = 0$. Either condition applied to ϕ_i gives the five S_3's of P_∞, the reference pentahedron in the x coördinate system. These S_3's combined three at a time give the ten lines† which in terms of y_i and z_j are

	q_1'	q_2'	q_3'	q_4'	q_5'
	$y_0 = 0$	$y_0 = 0$	$y_1 = 0$	$y_1 = 0$	$y_0 = 0$
Q_5'	$y_2 = 0$	$y_1 + z_1 = 0$	$y_2 + z_2 = 0$	$y_2 - z_2 = 0$	$y_2 - z_2 = 0$
	$z_2 = 0$	$y_2 + z_2 = 0$	$y_1 - z_1 = 0$	$z_1 = 0$	$y_1 - z_1 = 0$
	q_1''	q_2''	q_3''	q_4''	q_5''
	$y_0 = 0$	$y_0 = 0$	$y_1 + z_1 = 0$	$y_2 = 0$	$y_0 = 0$
Q_5''	$y_1 = 0$	$y_1 + z_1 = 0$	$y_2 = 0$	$z_2 = 0$	$y_2 + z_2 = 0$
	$z_1 = 0$	$y_2 - z_2 = 0$	$z_2 = 0$	$y_1 - z_1 = 0$	$y_1 - z_1 = 0$

The double points of the degenerate curves are the vertices of P_∞. They project into the vertices of a triangle, Δ_∞, the reference triangle in the fixed

* It is noteworthy that the transformations T_1' and S_1' on the ternary variables y_i are contragredient to the corresponding transformations on three quadratics in the binary domain, e. g., if $A_0 = -z_1 z_2$, $A_1 = z_2^2$, $A_2 = -z_1^2$ the result of operating with T' is

$$A_0' = A_0 + A_1 + A_2, \qquad A_1' = 2A_0 + (\epsilon^2 + \epsilon^3) A_1 + (\epsilon + \epsilon^4) A_2,$$
$$A_2' = 2A_0 + (\epsilon + \epsilon^4) A_1 + (\epsilon^2 + \epsilon^3) A_2.$$

The result of operating with S_1' is

$$A_0' = A_0, \qquad A_1' = \epsilon A_1, \qquad A_2' = \epsilon^4 A_2.$$

See Klein *Vorlesungen über das Ikosaeder*, pp. 211 ff.

† The lines are ordered as is indicated by the subscripts. The arrangement given is the unique one in which the ten lines form two skew pentagons so that all the lines are included, none are repeated. It was determined by a consideration of corresponding points and lines under C_1'.

plane π of C_1'. The vertices e_1 and e_2 of Δ_∞ are doubly covered, e_0 singly covered. The lines q_1' and q_1'' are unaltered by C_1', while the remaining lines of each degenerate curve are interchanged by pairs. The lines q_1' and q_1'' cut π in the points e_1 and e_2 which are therefore the projections of q_1' and q_1'' respectively. The table shows the entire projection of Q_5' and Q_5''.

q_1'	projects	into	e_1,	q_1''	projects	into	e_2,
q_2'	"	"	ϵ_0,	q_2''	"	"	ϵ_0,
q_3'	"	"	ϵ_1,	q_3''	"	"	ϵ_2,
q_4'	"	"	ϵ_1,	q_4''	"	"	ϵ_2,
q_5'	"	"	ϵ_0,	q_5''	"	"	ϵ_0.

Now the triangle Δ_∞ is also obtained by taking the intersections of the S_3's of P_∞ with π. Since P_i can be obtained from P_∞ by operating with $\boldsymbol{TS^i}$, the remaining five triangles, Δ_i, in π can be found by operating on Δ_∞ with the substitutions in the plane which correspond to $\boldsymbol{TS^i}$ of S_4.

It has been seen that $z = \tau$ gives the tangent plane at the point where the fixed line L of C_1' cuts Q_5 (vide § 18). For Q_5' and Q_5'', L cuts q_1' and q_1''. But $\tau_2 = z_2 = 0$ gives in π the point e_1 and $\tau_1 = z_1 = 0$ gives in π the point e_2 (vide (29)). Hence the modulus attached to Q_5' is $\tau_2 = 0$, to Q_5'' is $\tau_1 = 0$.

In the triangle Δ_∞ the side ϵ_0 is isolated. This isolated side is a double flex tangent of the sextic (32). The parameters of the two flexes are $z_1 = 0$, $z_2 = 0$. The parameters of all the flexes of (32) are given by the icosahedral form α_z^{12}. The remaining double flex tangents are derived from $y_0 = 0$ by operating with $\boldsymbol{T' S'^{\nu}}$ $(\nu = 0, \cdots, 4)$. The result is

$$y_0 + \epsilon^{2\nu} y_1 + \epsilon^{3\nu} y_2 = 0.$$

The six double flex tangents are isomorphic with the six lines of Klein's icosahedral configuration in the plane* (vide Klein, *Vorlesungen über das Ikosaeder*, pp. 211 ff). The invariant conic of G_{60} in the plane is the conic on the flexes

$$y_0^2 + y_1 y_2 = 0.$$

It cuts out on the sextic the icosahedral points.

In the plane π lie three double points of the involution C_1'. They can be obtained by setting $z_1 = z_2 = 0$ in ϕ_0', ϕ_1', ϕ_2'. If from the forms ϕ_i' thus modified, τ_1^2, $\tau_1 \tau_2$, τ_2^2 are eliminated, a sextic is obtained as the locus of the three fixed points of C_1', lying in π, for the whole family of Q_5's. Its equation is

* The rest of the configuration is not particularly interesting in connection with Q_5 so it is not given. A more complete discussion of the sextic (32) is given by R. M. Winger, *Self-projective rational sextics*, American Journal of Mathematics, vol. 38, January, 1916.

(33) $$y_0(y_1^5 + y_2^5) + 2y_0^2 y_1^2 y_2^2 - 8y_0^4 y_1 y_2 - y_1^3 y_2^3 = 0.$$

In § 18 the point $z = \tau$ on the doubly-covered conic (31) was accounted for. Its locus was (32), the locus of the fixed point of C_1' found on L. The fixed points of C_1' found on π are given by $\alpha_\tau^9 \alpha_z^3 = 0$, and their locus has now been found to be (33).

22. Return for a moment to a consideration of the projection of Q_5 by a pencil of S_3's on π,

(30) $$t_2 z_1 = t_1 z_2$$

to determine what happens when t is a root of $\alpha_t^{12} = 0$.

The fixed planes of the involutory collineations of G_{60} in S_4 lie by fives on the five vertices of P_∞. There are as many such arrangements as there are pentahedra. Hence there are thirty points on each of which is a fixed plane. In every plane there are six points. *The twenty-four points other than the six in the plane* $z_1 = 0, z_2 = 0$, *lie by pairs on* S_3*'s in the pencil* (30) *where* t *assumes the values of the roots of* $\alpha_t^{12} = 0$.